Convexity preserving interpolatory subdivision with conic precision

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« Quadratic objects » in CAD

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thinkID

Giering/Seybold: Konstruktive Ingenieurgeometrie

Catia
Convexity preserving interpolatory subdivision with conic precision
Conic reproduction
Related work

- **Convexity preserving subdivision:**
  - N. Dyn, Three families of nonlinear subdivision schemes, in Topics in Multivariate Approximation and Interpolation, 2005
  - F. Kuijt, R. Van Damme, Shape preserving interpolatory subdivision schemes for nonuniform data, J. Approx. Theory, 2002
  - M. Marinov, N. Dyn, D. Levin, *Geometrically controlled 4-point interpolatory schemes*, in Advances in Multiresolution for Geometric Modelling, Springer-Verlag, 2005

- **Conic reproducing subdivision:**
  - N. Aspert, T. Ebrahimi, P. Vanderghynst, Non-linear subdivision using local spherical coordinates, CAGD, 2003
  - C. Beccari, G. Casciola, L. Romani, Shape-controlled interpolatory ternary subdivision, submitted
Overview of the presentation

1. Idea for totally convex data
2. Adaptation for non-convex data
3. Proofs of the properties
   – Convexity preservation
   – Conic reproduction
   – C0 and C1 continuity
4. Examples
1. Idea: For every given point $p$ construct a line $l$ through $p$, yielding a **convex delimiting polygon** for the **new points**

\[ M_{33} := Q_3 \wedge (M_{15} \wedge (A \wedge B)) \]
Generation of the convex delimiting polygon

Construction of a (tangent) line $L_i$ in every point $P_i$

Closed polygon

$T_i = L_i \wedge L_{i+1}$

Open polygon
Basics from projective geometry

$$cr(U, P, X, T) = -1$$

harmonic cross ratio
Construction of a new point according to an angle criterion

\[ \alpha_{ji}^i = \min_j \alpha_j^i \]
Construction of a new point $U_i$

$$cr(U_i, P_j, X_i, T_i) = -1$$
Adaptation for non-convex data

Segmentation

Colinear points

Inflection points

Convex junction points
Adaptation for non-convex data

Inflection points

\[ \gamma_{2k_i}^k = \max\{\gamma_{l,2k_i}^k, \gamma_{r,2k_i}^k\} \Rightarrow g_{2k_i}^k \]

\[ l_i^0 = \lambda_{i}^0 l_{i,i}^0 + \mu_{i}^0 l_{r,i}^0 \]

Initial tangent

k-th iteration tangent
Adaptation for non-convex data

Convex junction points

k-th iteration tangent

\[ l^k_{2k_i} = \lambda^k_{2k_i} l^k_{l,2k_i} + \mu^k_{2k_i} l^k_{r,2k_i} \]

\[ \lambda^k_{2k_i} + \mu^k_{2k_i} = 1, \quad \lambda^k_{2k_i}, \mu^k_{2k_i} > 0 \]
Properties of the subdivision scheme

• convexity preservation
• conic reproduction

• continuity:
  - C0 continuity
  - C1 continuity
Properties of the subdivision scheme

- \( C^0 \) continuity

If \( p^0 \) convex \( \Rightarrow \) \( p^k \) convex and bounded by \( q^{k_0} \)

\( p^k \subset p^{k+1} \), \( \{ p^k \} \) .... monotone and bounded
Properties of the subdivision scheme

- C0 continuity

If $p^0$ convex $\Rightarrow$ $p^k$ convex and bounded by $q^{k_0}$

$p^k \subset p^{k+1}$, $\{p^k\}$ .... monotone and bounded

$\Rightarrow$ (Dyn, Levin, Liu CAD 1992)$\Rightarrow$

$$\lim_{k \to \infty} p^k$$ exists and is a $C^0$ convex curve

If $p^0$ non-convex $\Rightarrow$ (by construction)$\Rightarrow$

$$\lim_{k \to \infty} p^k$$ exists and is a $C^0$ curve
Properties of the subdivision scheme

- C1 continuity

\[ n_i^k = (\cos(\theta_i^k), \sin(\theta_i^k)) \perp (p_{i+1}^k - p_i^k) \]

\[ \theta^k : \mathbb{R} \to \mathbb{R} : \theta^k(\tilde{t}_i^k) = \theta_i^k \]
Properties of the subdivision scheme

- C1 continuity

\[ \mathbf{n}_i^k = (\cos(\theta_i^k), \sin(\theta_i^k)) \perp (\mathbf{p}_{i+1}^k - \mathbf{p}_i^k) \]

\[ \theta^k : \mathbb{R} \rightarrow \mathbb{R}, \quad \theta^k(\tilde{t}_i^k) = \theta_i^k \]

=> numerically:

\[ \lim_{k \rightarrow \infty} \theta^k = \theta \quad \text{continuous} \]
Properties of the subdivision scheme

- **C1 continuity**

\[ \tilde{p}_i^{k,[1]} = \frac{p_{i+1}^k - p_i^k}{\tilde{t}_{i+1}^k - \tilde{t}_i^k} = (- \sin(\theta_i^k), \cos(\theta_i^k)) \]

\[ \Delta_{k,[1]} = \sup_{i \in \mathbb{Z}} \| \tilde{p}_{i+1}^{k,[1]} - \tilde{p}_i^{k,[1]} \|_2 \]
Examples: uniformly spaced data
Examples: non-uniformly spaced data
Examples: conics
Examples: curvature combs

Dyn, Floater, Hormann, CAGD 2009
Sabin, Dodgson, Proc. Tromso 2005
Examples: curvature combs

Albrecht, Romani


Dyn, Floater, Hormann, CAGD 2009

Sabin, Dodgson, Proc. Tromso 2005
Examples: general, non-convex data

Data: courtesy of think3
Theoretical results:
- Convexity preservation
- Conic reproduction
- C1 limit curve