Pricing pension plans based on average salary under jump-diffusion models

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Outline

1. Introduction and objectives
2. Mathematical modeling
3. Numerical methods
4. Numerical results
5. Conclusions
General classification of pension plans

- **Pension plans with defined benefits**: benefit on retirement is determined by a set formula and not depending on investment returns.

- **Pension plans with defined contributions**: contributions are paid into an individual account for each member. Contributions are invested and the returns of the investment are credited to the individual’s account.
Some general introductory references

Aim of the work

- Benefits of the pension plan understood as the value of the liabilities of the plan with and active member
- Modeling the presence of jumps in the underlying salary
- Pricing the benefits of a pension plan where the retirement depends on the average salary of the member
- PIDE approach (option pricing analogy)
- Monte Carlo simulation
- Early retirement: free boundary problem
- Appropriate numerical methods for solving the model
Outline

1. Introduction and objectives

2. Mathematical modeling

3. Numerical methods

4. Numerical results

5. Conclusions
 Characteristics of the pension plan

- The retirement benefits of the plan depend on the average salary of the last $n_y$ years.
- Two approaches: with and without early retirement.
- Three possible situations of a member:
  - active
  - dead
  - withdrawn or resigned.
- Transition intensities from active state to dead or resign ones influence the value of benefits of the plan at retirement.

Previous work in the absence of jumps


Stochastic model for salary evolution (I)

$t_0$ : Entry age of a member into the pension plan
$t$ : Time since the member enters the plan
$S_t$ : Salary at time $t$ of a member entering with age $t_0$

\[
dS_t = \alpha(t, S_t) \, dt + \sigma(t, S_t) \, dZ_t + d\left(\sum_{i=1}^{N_t} Y_i\right)
\]

where

$\alpha(t, S_t)$ : Growth rate of the salary $S$ at age $t + t_0$
$\sigma(t, S_t)$ : Volatility of the salary $S$ at age $t + t_0$
$Z_t$ : Wiener process
$(N_t)_{t \geq 0}$ : Poisson process with parameter $\tilde{\lambda}$
$(Y_i)$ : sequence of i.i.d random variables
Stochastic model for salary evolution (II)

Merton’s jump-diffusion model

- $(Y_i) \sim LN(\mu, \gamma^2)$
- with the density

$$\nu(y) = \frac{1}{y\gamma\sqrt{2\pi}} \exp\left(-\frac{(\log y - \mu)^2}{2\gamma^2}\right)$$

R. C. Merton, Option pricing when underlying stock returns are discontinuous, J. Finan. Econ., 3 (1976), 125-144.
New variable associated to average salary

- The plan is indicated to the last $n_y$ year average salary
- Analogously to Asian options we introduce

$$I_t = \int_0^t g(\tau, S_\tau) \, d\tau$$

where (for a given function $h$):

$$g(t, S) = \begin{cases} 
0 & 0 \leq t < T_r - n_y \\
h(S) & T_r - n_y \leq t \leq T_r
\end{cases}$$

- The variation of $I$ from $t$ to $t + dt$ is given by

$$dl = I_{t+dt} - I_t = \int_t^{t+dt} g(\tau, S_\tau) \, d\tau = g(t, S_t) \, dt$$
Benefits of the pension plan

- \( V = V(t, S_t, l_t) \): Value of the benefits of the pension plan
- \( r(t) \): Risk free interest rate at time \( t \)
- \( \mu_d(t) \): Transition intensity from active to dead
- \( \mu_w(t) \): Transition intensity from active to withdrawn
- \( A_i(t, S_t) - V(t, S_t, l_t) \): Sum-at-risk associated to death \((i = d)\) or withdrawal \((i = w)\)

In deterministic setting, variation of the value of the benefits from time \( t \) to time \( t + dt \):

\[
dV = r(t)Vdt + \sum_{i=d,w} \mu_i(t) (A_i(t, S_t) - V(t, S_t, l_t)) \, dt
\]

(Thiele’s differential equation)
Let $\Omega = (0, +\infty)^2$

**PIDE in $(0, T_r) \times \Omega$**

\[
\frac{\partial_t V}{\partial t} + \beta \frac{\partial_S V}{\partial S} + g \frac{\partial_I V}{\partial I} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} V - (\mu_d + \mu_w + r) V \\
+ \int_0^\infty \tilde{\lambda} \left[ V(t, Sy, I) - V(t, S, I) - S(y - 1) \frac{\partial_S V(t, S, I)}{\partial S} \right] \nu(y) dy \\
= -\mu_d A_d - \mu_w A_w
\]
PIDE model for a pension plan based on average salary under jump-diffusion (II)

- Considering that:

\[
\int_0^\infty \nu(y)dy = 1 \quad \text{and} \quad E[Y_i] = \int_0^\infty y\nu(y)dy = e^{\mu+\gamma^2/2}
\]

**PIDE in \((0, T_r) \times \Omega\)**

\[
\partial_t V + (\beta - \tilde{\lambda}\kappa S) \partial_S V + g \partial_I V + \frac{1}{2} \sigma^2 \partial_{SS} V - \left(\mu_d + \mu_w + r + \tilde{\lambda}\right) V \\
+ \tilde{\lambda} \int_0^\infty V(t, Sy, l)\nu(y)dy = -\mu_d A_d - \mu_w A_w
\]

where \(\kappa = e^{\mu+\gamma^2/2} - 1\)
PIDE model for a pension plan based on average salary under jump-diffusion (III)

Let $\beta = \theta S$ and $\sigma(t, S_t) = \sigma S$, with $\theta > 0$ and $\sigma > 0$

No early retirement: Cauchy problem

$$\begin{cases} 
\mathcal{L}_j[V] = f, & \text{in } (0, T_r) \times \Omega, \\
V(T_r, S, I) = \frac{a}{n_y} I, & (S, I) \in \Omega
\end{cases}$$

Early retirement: complementarity problem

$$\begin{cases} 
\max\{\mathcal{L}_j[V] - f, \Psi - V\} = 0, & \text{in } (0, T_r) \times \Omega, \\
V(T_r, S, I) = \frac{a}{n_y} I, & (S, I) \in \Omega
\end{cases}$$
PIDE model for a pension plan based on average salary under jump-diffusion (IV)

where

$$\mathcal{L}_j[V] = \partial_t V + (\theta - \tilde{\lambda} \kappa) S \partial_S V + g \partial_I V + \frac{\sigma^2 S^2}{2} \partial_{SS} V$$

$$- \left( r + \mu_d + \mu_w + \tilde{\lambda} \right) V + \tilde{\lambda} \int_0^\infty V(t, Sy, I) \nu(y) dy,$$

$$f(t, S, I) = - (\mu_d \alpha_d + \mu_w \alpha_w) S$$

and

$$\psi(t, S, I) = \begin{cases} 0 & \text{if } t < T_0, \\
\left(1 - \frac{T_r - t}{T_r - T_0}\right) \frac{al}{t - (T_r - n_y)} & \text{if } t \geq T_0 \end{cases}$$
Divergence form in the unbounded domain (I)

Change of time variable: $\tau = T_r - t$, Spatial variables: $x_1 = S$, $x_2 = l$

$$\tilde{L}_j[V] = \partial_\tau V + \vec{v} \cdot \nabla V - \text{Div}(A \nabla V) + l^2 V - \tilde{\lambda} \int_0^\infty V(\tau, x_1 y, x_2) \nu(y) dy$$

$$A(x_1, x_2) = \begin{pmatrix} \frac{1}{2} \sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\vec{v}(\tau, x_1, x_2) = \begin{pmatrix} (\sigma^2 - \theta + \tilde{\lambda} \kappa) x_1 \\ -g(T_r - \tau, x_1) \end{pmatrix}$$

$$l(\tau, x_1, x_2) = r + \mu_d + \mu_w + \tilde{\lambda}$$
Divergence form in the unbounded domain (II)

No early retirement: Cauchy problem

\[
\begin{align*}
\bar{L}_j[V] &= f, & \text{in } (0, T_r) \times \Omega, \\
V(0, x_1, x_2) &= \frac{a}{ny} x_2, & (x_1, x_2) \in \Omega
\end{align*}
\]

Early retirement: complementarity problem

\[
\begin{align*}
\max\{\bar{L}_j[V] - f, \bar{\Psi} - V\} &= 0, & \text{in } (0, T_r) \times \Omega, \\
V(0, x_1, x_2) &= \frac{a}{ny} x_2, & (x_1, x_2) \in \Omega
\end{align*}
\]

where

\[f(\tau, x_1, x_2) = (\mu_d \alpha_d + \mu_w \alpha_w)x_1 \quad \text{and} \quad \bar{\Psi}(\tau, x_1, x_2) = \Psi(T_r - \tau, x_1, x_2)\]
Main difficulties in the numerical solution

- Unbounded domain in salary and accumulated salary directions
- Unbounded domain of integration in the integral term
- Diffusion matrix is degenerated (convection dominated problem)

Localization + boundary conditions
Localization
Higher order Lagrange/Galerkin methods
Main difficulties in the numerical solution

- Unbounded domain in salary and accumulated salary directions
- Localization + boundary conditions
- Unbounded domain of integration in the integral term
- Localization
- Diffusion matrix is degenerated
  (convection dominated problem)
- Higher order Lagrange/Galerkin methods
Let $x_1^\infty$ and $x_2^\infty$ large enough.

Let $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$ be the bounded domain.

Let $\Gamma = \partial \Omega = \bigcup \Gamma_1^+ \cup \bigcup \Gamma_2^+ \cup \bigcup \Gamma_1^- \cup \bigcup \Gamma_2^-$. 
Integral term localization

Change of variables: $\bar{x}_1 = \log(x_1)$ and $\eta = \log(y)$

$$\int_{0}^{\infty} V(\tau, x_1y, x_2)\nu(y)dy \approx \int_{\eta_{\text{min}}}^{\eta_{\text{max}}} V(\tau, \bar{x}_1 + \eta, x_2)\bar{\nu}(\eta)d\eta$$

where

$$\bar{\nu}(\eta) = \frac{1}{\gamma \sqrt{2\pi}}\exp\left(-\frac{(\eta - \mu)^2}{2\gamma^2}\right)$$

and if we consider the domain $\Omega = [0, x_1^\infty] \times [0, x_2^\infty]$ and the discrete grid $0 = x_{10}, x_{11}, \cdots, x_{1q} = x_1^\infty$ then $\eta_{\text{min}} = \log(x_{11})$ and $\eta_{\text{max}} = \log(x_{1q})$

Formulation in the bounded domain (I)

- **No early retirement**
  Find $V : [0, T_r] \times \Omega \rightarrow \mathbb{R}$, such that
  \[
  \partial_\tau V - \text{Div}(A \nabla V) + \vec{v} \cdot \nabla V + IV - \lambda \int_{\eta_{min}}^{\eta_{max}} \bar{V}(\tau, \bar{x}_1 + \eta, x_2)\bar{v}(\eta) d\eta = f \quad \text{in} \ (0, T_r) \times \Omega
  \]

- **Early retirement**
  Find $V$ and $P : [0, T_r] \times \Omega \rightarrow \mathbb{R}$, satisfying
  \[
  \partial_\tau V - \text{Div}(A \nabla V) + \vec{v} \cdot \nabla V + IV - \lambda \int_{\eta_{min}}^{\eta_{max}} \bar{V}(\tau, \bar{x}_1 + \eta, x_2)\bar{v}(\eta) d\eta + P = f \quad \text{in} \ (0, T_r) \times \Omega
  \]
  the complementarity conditions
  \[
  V \geq \bar{\Psi}, \quad P \leq 0, \quad (V - \bar{\Psi})P = 0 \quad \text{in} \ (0, T_r) \times \Omega
  \]
Formulation in the bounded domain (II)

jointly with the initial and boundary conditions

\[ V(0, x_1, x_2) = \frac{a}{n_y} x_2 \quad \text{in} \quad \Omega \]
\[ \frac{\partial V}{\partial x_1} = 0 \quad \text{on} \quad (0, T_r) \times \Gamma_1^+ \]
\[ \frac{\partial V}{\partial x_2} = \frac{a}{n_y} \quad \text{on} \quad (0, T_r) \times \Gamma_2^+ \]

Time discretization: higher order characteristics

Velocity field:

$$\vec{v} = \begin{pmatrix} (\sigma^2 - \theta + \tilde{\lambda}\kappa)x_1 \\ -\bar{g}(\tau, x_1) \end{pmatrix}$$

$$\bar{g}(\tau, x_1) = \begin{cases} 0 & \text{if } \tau > n_y \\ k_1x_1 & \text{if } \tau \leq n_y \end{cases}$$

Characteristics curves through $x = (x_1, x_2)$ at time $\tau_{n+1}$:

$$X_e(x, \tau_{n+1}; s)$$

$$\frac{\partial}{\partial s} X_e(x, \tau_{n+1}; s) = \vec{v}(X_e(x, \tau_{n+1}; s)), \quad X_e(x, \tau_{n+1}; \tau_{n+1}) = x$$
Final value problem can be exactly solved:

- If \((\sigma^2 - \theta + \tilde{\lambda}\kappa) \neq 0\) then \([X^n_e]_1(x) = x_1 \exp((\theta - \sigma^2 - \tilde{\lambda}\kappa)\Delta\tau)\) and

\[
[X^n_e]_2(x) = \begin{cases} 
x_2 & \text{if } n\Delta\tau > n_y \\
k_1x_1 \frac{k_1x_1}{\sigma^2 - \theta + \tilde{\lambda}\kappa} (1 - \exp((\theta - \sigma^2 - \tilde{\lambda}\kappa)\Delta\tau)) + x_2 & \text{if } n\Delta\tau \leq n_y
\end{cases}
\]

- If \((\sigma^2 - \theta + \tilde{\lambda}\kappa) = 0\) then \([X^n_e]_1(x) = x_1\) and

\[
[X^n_e]_2(x) = \begin{cases} 
x_2 & \text{if } n\Delta\tau > n_y \\
k_1x_1\Delta\tau + x_2 & \text{if } n\Delta\tau \leq n_y
\end{cases}
\]

where \(X^n_e(x) := X_e(x, \tau^{n+1}; \tau^n)\)
Higher order characteristics

Spatial discretization:
Characteristics-Crank-Nicolson scheme

Time step $\Delta \tau = \frac{T_r}{N}$
Time meshpoints $\tau_n = n \Delta \tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$.

Characteristics for material derivative approximation:

$$\frac{DV}{D\tau} = \frac{V^{n+1} - V^n \circ X^n_e}{\Delta \tau}$$

where $X^n_e(x) := X_e(x, \tau^{n+1}; \tau^n)$

Crank-Nicolson around $(X_e(x, \tau_{n+1}; \tau), \tau)$ for $\tau = \tau_{n+\frac{1}{2}}$
Char-CN-Explicit scheme for time discretization

For \( n=0,\ldots,N-1 \), find \( V^{n+1} \) such that:

\[
\frac{V^{n+1}(x) - V^n(x_e^n(x))}{\Delta \tau} - \frac{1}{2} \text{Div}(A \nabla V^{n+1})(x) - \frac{1}{2} \text{Div}(A \nabla V^n)(x_e^n(x)) \\
+ \frac{1}{2}(IV^{n+1})(x) + \frac{1}{2}(IV^n)(x_e^n(x)) - \bar{\lambda} \int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \bar{V}^n(\bar{x}_1 + \eta, x_2) \bar{v}(\eta) d\eta \\
= \frac{1}{2}(f^{n+1})(x) + \frac{1}{2}(f^n)(x_e^n(x)),
\]

where \( \bar{V}^n(\bar{x}_1 + \eta, x_2) = V^n(e^{\bar{x}_1+\eta}, x_2) \)

Let $X: \bar{\Omega} \rightarrow X(\bar{\Omega}), X \in C^2(\bar{\Omega})$, be a vectorial invertible map. Let $F = \nabla X$ and assume that $F^{-1} \in C^1(\bar{\Omega})$. Then, we have:

$$\int_{\Omega} \text{div} w(X(x)) \psi(x) \, dx = \int_{\Gamma} F^{-T}(x)n(x) \cdot w(X(x))\psi(x) \, dA_x$$

$$- \int_{\Omega} F^{-1}(x)w(X(x)) \cdot \nabla \psi(x) \, dx$$

$$- \int_{\Omega} \text{Div} F^{-T} \cdot w(X(x))\psi(x) \, dx,$$

where $w \in H^1(X(\Omega))$ is a vectorial map and $\psi \in H^1(\Omega)$ is a scalar function.
Some intermediate calculus: Green formulas

\[
\int_\Omega \frac{V^{n+1} - V^n \circ X_e^n}{\Delta \tau} \psi d\mathbf{x} + \frac{1}{2} \int_\Omega A \nabla V^{n+1} \nabla \psi d\mathbf{x} +
\]
\[
+ \frac{1}{2} \int_\Omega (F^n_e)^{-1} (A \nabla V^n)(X^n_e(x)) \nabla \psi d\mathbf{x} + \frac{1}{2} \int_\Omega (lV^{n+1})(\mathbf{x}) \psi d\mathbf{x}
\]
\[
+ \frac{1}{2} \int_\Omega (lV^n)(X^n_e(\mathbf{x})) \psi d\mathbf{x} = \frac{1}{2} \int_\Omega (f^{n+1})(\mathbf{x}) \psi d\mathbf{x} + \int_\Omega \frac{1}{2} (f^n)(X^n_e(\mathbf{x})) \psi d\mathbf{x}
\]
\[
+ \frac{1}{2} \int_{\Gamma} \mathbf{n} \cdot A \nabla V^{n+1} \psi dA_{\mathbf{x}} + \frac{1}{2} \int_{\Gamma} (F^n_e)^{-T} \mathbf{n} \cdot (A \nabla V^n)(X^n_e(\mathbf{x})) \psi dA_{\mathbf{x}}
\]
\[
+ \tilde{\lambda} \int_\Omega \int_{\eta_{\min}}^{\eta_{\max}} \tilde{V}^n(\bar{x}_1 + \eta, \bar{x}_2) \tilde{v}(\eta) d\eta \psi d\mathbf{x}
\]

where \( F^n_e = \nabla X^n_e \)
Some intermediate calculus: Boundary terms

- We have $\vec{n} \cdot A \nabla V^{n+1} = 0$ on $\Gamma_{1}^{-} \cup \Gamma_{2}^{-} \cup \Gamma_{2}^{+}$
- We impose $\frac{\partial V}{\partial x_1} = g_1 = 0$ on $\Gamma_{1}^{+}$. So, we have:

$$\int_{\Gamma} \vec{n} \cdot A \nabla V^{n+1} \psi \, dA_x = 0$$
Some intermediate calculus: Boundary terms

\[ \int_{\Gamma} (F^n_e)^{-T} \vec{n} \cdot (A \nabla V^n)(X^n_e(L)) \psi dA_x = \int_{\Gamma} \tilde{g}^n \psi dA_x \]

where \( \tilde{g}^n \) is defined as follows:

\[
\tilde{g}^n(x) := \begin{cases} 
0 & \text{on } \Gamma_{1,-} \\
- \left[ (F^n_e)^{-T} \right]_{12}(x) a_{11}(X^n_e(x)) \frac{\partial V}{\partial x_1}(X^n_e(x)) & \text{on } \Gamma_{2,-} \\
\left[ (F^n_e)^{-T} \right]_{12}(x) a_{11}(X^n_e(x)) \frac{\partial V}{\partial x_1}(X^n_e(x)) & \text{on } \Gamma_{2,+} \\
\left[ (F^n_e)^{-T} \right]_{11}(x) a_{11}(X^n_e(x)) g^1_n(X^n_e(x)) & \text{on } \Gamma_{1,+} 
\end{cases}
\]
Variational formulation

For \( n=0,\ldots,N-1 \), find \( V^{n+1} \in H^1(\Omega) \) such that, \( \forall \psi \in H^1(\Omega) : \)

\[
\int_{\Omega} V^{n+1}(x) \psi \, dx + \frac{\Delta \tau}{2} \int_{\Omega} (A \nabla V^{n+1})(x) \nabla \psi \, dx + \frac{\Delta \tau}{2} \int_{\Omega} l V^{n+1}(x) \, dx = \\
\int_{\Omega} V^n(X_e^n(x)) \psi \, dx - \frac{\Delta \tau}{2} \int_{\Omega} (F^n_e)^{-1}(x)(A \nabla V^n)(X_e^n(x)) \nabla \psi \, dx - \\
\frac{\Delta \tau}{2} \int_{\Omega} l V^n(X_e^n(x)) \psi \, dx + \frac{\Delta \tau}{2} \int_{\Gamma} \tilde{g}^n(x) \psi \, dA_x + \\
\frac{\Delta \tau}{2} \int_{\Omega} f^{n+1}(x) \psi \, dx + \frac{\Delta \tau}{2} \int_{\Omega} f^n(X_e^n(x)) \psi \, dx + \\
\Delta \tau \tilde{\lambda} \int_{\Omega} \int_{\eta_{\text{max}}}^{\eta_{\text{min}}} \bar{V}^n(\bar{x}_1 + \eta, x_2) \bar{v}(\eta) \, d\eta \psi \, dx
\]
Spatial Discretization: finite elements

For $V_h^0 \in V_h$, find $V_h = \{V_h^n\}_{n=1}^N \in [V_h]^N$ such that

$$\frac{1}{\Delta \tau} < D_E^{n+1}[V], \psi_h > + < M^n[V], \psi_h > = < N^n, \psi_h >$$

for all $\psi_h \in V_h$ and $n = 0, ..., N - 1$,

$$V_h = \{\phi_h \in C^0(\overline{\Omega}) : \phi_h|_T \in Q_2, \forall T \in \tau_h\}$$
Integral term approximation

Composite trapezoidal rule with $m+1$ points

\[
\int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \tilde{V}^n(\bar{x}_1 + \eta, x_2) \tilde{v}(\eta) \, d\eta \approx \frac{h}{2} \left[ \tilde{V}^n(\bar{x}_1 + \eta_{\text{min}}, x_2) \tilde{v}(\eta_{\text{min}}) + \tilde{V}^n(\bar{x}_1 + \eta_{\text{max}}, x_2) \tilde{v}(\eta_{\text{max}}) + 2 \sum_{j=1}^{m-1} \tilde{V}^n(\bar{x}_1 + k_j, x_2) \tilde{v}(k_j) \right]
\]

where $k_j = \eta_{\text{min}} + jh$ for $j = 1, \ldots, m - 1$ and $h = \frac{\eta_{\text{max}} - \eta_{\text{min}}}{m}$. 
Augmented Lagrangian Active Set (ALAS) algorithm

Notation:

\[ N := 1, 2, \ldots, N_{\text{dof}}, \quad N_{\text{dof}} := \text{dof of FEM} \]

For each time \( \tau_n \):

Find \( V^n_h, P^n_h \) and a decomposition \( N = J^n \cup I^n \) such that

\[
M_h V^n_h + P^n_h = b^{n-1}_h,
\]

\[
[P^n_h]_j + \beta \left[ V^n_h - \bar{\Psi} \right]_j \leq 0 \quad \forall j \in J^n,
\]

\[
[P^n_h]_i = 0 \quad \forall i \in I^n,
\]

for any \( \beta > 0 \).

- \( I^n \): discrete inactive set (hold region) at time \( \tau_n \)
- \( J^n \): discrete active set (exercise region) at time \( \tau_n \)
Augmented Lagrangian Active Set (ALAS) algorithm

Notation:

\[ N := 1, 2, \ldots, N_{dof}, \quad N_{dof} := \text{dof of FEM} \]

For each time \( \tau_n \):

Building sequences

\[ V_{h,m}^n \rightarrow V_h^n \]
\[ P_{h,m}^n \rightarrow P_h^n \]
\[ I_m^n \rightarrow I^n \]
\[ J_m^n \rightarrow J^n \]

- \( I^n \): discrete inactive set (hold region) at time \( \tau_n \)
- \( J^n \): discrete active set (exercise region) at time \( \tau_n \)
Augmented Lagrangian Active Set (ALAS) algorithm

1. Initialize $V^n_{h,0}$ and $P^n_{h,0} \leq 0$. Choose $\beta > 0$. Set $m = 0$.

2. Compute

\[
Q^n_{h,m} = \min \left\{ 0, P^n_{h,m} + \beta \left( V^n_{h,m} - \bar{\Psi} \right) \right\},
\]

\[
J^n_m = \left\{ j \in N, \left[ Q^n_{h,m} \right]_j < 0 \right\},
\]

\[
I^n_m = \left\{ i \in N, \left[ Q^n_{h,m} \right]_i = 0 \right\}.
\]

3. If $m \geq 1$ and $J^n_m = J^n_{m-1}$ then convergence and stop.

4. Let $V$ and $P$ be the solution of the linear system (complete)

\[
M_h V + P = b^{n-1},
\]

\[
P = 0 \text{ on } I^n_m \text{ and } V = \bar{\Psi} \text{ on } J^n_m.
\]

Set $V^n_{h,m+1} = V$, $P^n_{h,m+1} = \min\{0, P\}$, $m = m + 1$.

Go to 2.
Augmented Lagrangian Active Set (ALAS) algorithm

1. Initialize $V_{h,0}^n$ and $P_{h,0}^n \leq 0$. Choose $\beta > 0$. Set $m = 0$.

2. Compute

$$Q_{h,m}^n = \min \left\{ 0, P_{h,m}^n + \beta \left( V_{h,m}^n - \bar{\Psi} \right) \right\},$$

$$J_m^n = \left\{ j \in N, \left[ Q_{h,m}^n \right]_j < 0 \right\},$$

$$I_m^n = \left\{ i \in N, \left[ Q_{h,m}^n \right]_i = 0 \right\}.$$

3. If $m \geq 1$ and $J_m^n = J_{m-1}^n$ then convergence and stop.

4. Let $V$ and $P$ be the solution of the linear system (reduced)

$$[M_h]_I [V]_I = [b^{n-1}]_I - [M_h]_J [\bar{\Psi}]_J,$$

$$[V]_J = [\bar{\Psi}]_J,$$

$$P = b^{n-1} - M_h V,$$

for $I = I_m^n$ and $J = J_m^n$. Go to 2.
Particular Features

- Convergence and monotonicity ($I^n_m \subset I^n_{m+1}$) if:
  - $M_h$ Stieljes matrix.
  - Suitable initialization.

Particular Features

- Convergence and monotonicity \( (I_m^n \subset I_{m+1}^n) \) if:
  - \( M_h \) Stieltjes matrix. Not for quadratic elements
  - Suitable initialization.

At each time step \( n \):

- Initialization

\[
V_{h,0}^n = \bar{\Psi} \quad \text{and} \quad P_{h,0}^n = \min(b^n - M_h V_{h,0}^n, 0).
\]

- Not assume monotonicity with respect to \( m \) for the sets \( \{I_m^n\} \).
Linear system solution

Complete

\[ M_h V + P = b^{n-1}, \]

\[ P = 0 \text{ on } l^n_m \text{ and } V = \bar{\Psi} \text{ on } J^n_m. \]

Reduced

\[ [M_h]_I \ [V]_I = [b^{n-1}]_I - [M_h]_{IJ} \ [\bar{\Psi}]_J, \]

\[ [V]_J = [\bar{\Psi}]_J, \]

\[ P = b^{n-1} - M_h V, \]

for \( I = l^n_m \) and \( J = J^n_m. \)
Particular Features

Linear system solution

Complete

\[ M_h V + P = b^{n-1}, \]
\[ P = 0 \text{ on } I^m_n \text{ and } V = \bar{\Psi} \text{ on } J^m_n. \]

Reduced

\[ [M_h]_{II} \begin{bmatrix} V \end{bmatrix}_I = \begin{bmatrix} b^{n-1} \end{bmatrix}_I - [M_h]_{IJ} \begin{bmatrix} \bar{\Psi} \end{bmatrix}_J, \]
\[ \begin{bmatrix} V \end{bmatrix}_J = \begin{bmatrix} \bar{\Psi} \end{bmatrix}_J, \]
\[ P = b^{n-1} - M_h V, \]

for \( I = I^m_n \) and \( J = J^m_n. \)

\( M_h \quad m \text{ and } n \text{ independent!} \)

\( [M_h]_{II} \quad m \text{ and } n \text{ dependent!} \)
Monte Carlo simulation (I)

Risk neutral dynamic of the salary

\[ dS_t = (\theta - \tilde{\lambda} \kappa) S_t dt + \sigma S_t dZ^Q_t + d \left( \sum_{i=1}^{N_t} Y_i \right) \]

- **Without early retirement**

\[ V(t, S, I) = E_Q \left[ e^{-(r+\mu_d+\mu_w)(T_r-t)} \frac{a}{n_y} I - \int_t^{T_r} e^{-(r+\mu_d+\mu_w)(u-t)} f(u, S_u) du \right] \]

- **With early retirement (Longstaff-Schwartz (LS) technique)**

\[ V(t, S, I) = \sup_{\tau \in T(t, T_r)} E_Q \left[ e^{-(r+\mu_d+\mu_w)(\tau-t)} \psi(\tau, S, I) - \int_t^{\tau} e^{-(r+\mu_d+\mu_w)(u-t)} f(u, S_u) du \right] \]

Monte Carlo simulation (II)

Simulating the salary with jumps at fixed dates:
0 = t_0 < t_1 < ... < t_m = T_r
We simulate X(t_j) = \log(S(t_j)) in the following way:

1. generate \( W \sim N(0, 1) \)
2. generate \( N \sim \text{Poisson}(\tilde{\lambda}(t_{j+1} - t_j)) \); if \( N = 0 \), set \( M = 0 \) and go to step 4
3. assuming \( \log Y_i \sim N(\mu, \gamma^2) \) and \( \sum_{i=1}^{n} \log Y_i \sim N(\mu n, \gamma^2 n) = an + b\sqrt{n}N(0, 1) \), then, generate \( W_2 \sim N(0, 1) \); set \( M = \mu N + \gamma \sqrt{NW_2} \)
4. set

\[
X(t_{j+1}) = X(t_j) + (\theta - \frac{1}{2} \sigma^2 - \tilde{\lambda}\kappa)(t_{j+1} - t_j) + \sigma \sqrt{t_{j+1} - t_j} W + M
\]

So, \( S(t_j) = e^{X(t_j)} \)
Outline

1 Introduction and objectives
2 Mathematical modeling
3 Numerical methods
4 Numerical results
5 Conclusions
$Q_2$ finite element meshes

<table>
<thead>
<tr>
<th>Mesh</th>
<th>N. Elem</th>
<th>N. Nodes</th>
</tr>
</thead>
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<td>Mesh 24</td>
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<td>Mesh 48</td>
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<td>Mesh 96</td>
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Table: FEM mesh data
Model data

<table>
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<th>Parameter</th>
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<tr>
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<td>$\mu_w$</td>
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</tr>
<tr>
<td>$\gamma$</td>
<td>0.45</td>
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Table: Pension plan data

- **PIDE model**: $\Omega = (0, 40) \times (0, 40)$
  - Parameter $\beta$ ALAS algorithm: 10000
  - Composite trapezoidal rule: $m = 50$
- **Monte Carlo simulation**: 7000 time steps per year and 50000 paths
Retirement benefits value without early retirement

<table>
<thead>
<tr>
<th>time steps</th>
<th>Mesh 12</th>
<th>Mesh 24</th>
<th>Mesh 48</th>
<th>Mesh 96</th>
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Table: Retirement benefits at time $t = 0$ and mesh point $(S,l) = (25,20)$. The 99% confidence interval with Monte Carlo simulation is $[2.582251, 2.651059]$. 
Retirement benefits value with early retirement

<table>
<thead>
<tr>
<th>time steps</th>
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Table: Retirement benefits at time $t = 0$ and mesh point $(S,I) = (25,20)$. The 99% confidence interval with LS simulation is $[2.589137, 2.667215]$
Pension plan values at $t = 0$ without early retirement

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<th>$\alpha$</th>
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<th>Monte Carlo ($S, l) = (1.2, 15)$</th>
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Table: Retirement benefit value for different parameters
### Pension plan values at $t = 38$ without early retirement

<table>
<thead>
<tr>
<th>$n_y$</th>
<th>$\sigma$</th>
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<th>(S, l) = (1.2, 15)</th>
<th>Monte Carlo</th>
<th>(S, l) = (1.2, 7.5)</th>
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<td>0.311833</td>
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<td>0.311550, 0.312009</td>
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**Table:** Retirement benefit value for different parameters
Comparison between the cases with and without early retirement

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<tr>
<th></th>
<th>((S, I) = (1.2, 15))</th>
<th>((S, I) = (2.4, 30))</th>
<th>((S, I) = (4, 10))</th>
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<tbody>
<tr>
<td><strong>V without ER</strong></td>
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<tr>
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<td>(0.586484, 0.587052)</td>
<td>(0.374073, 0.374953)</td>
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<td><strong>V with ER</strong></td>
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<td><strong>PIDE</strong></td>
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<td>0.739286</td>
<td>0.374635</td>
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<tr>
<td><strong>P with ER</strong></td>
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<tr>
<td><strong>PIDE</strong></td>
<td>(-1.645011 \times 10^{-5})</td>
<td>(-2.304657 \times 10^{-5})</td>
<td>0</td>
</tr>
</tbody>
</table>

Table: Retirement benefits under a jump diffusion process without and with early retirement (ER) at different points when time \(t = 38\). Computed values with (PIDE), Monte Carlo (MC) and Longstaff-Schwartz (LS) and Multiplier (P).
Conclusions

- A PIDE model for valuing a defined benefit pension plan indicated to the average salary under a jump diffusion process is proposed.
- The salary dynamic is governed by Merton’s jump-diffusion process.
- No early retirement: initial boundary problem associated with a parabolic integro-differential operator.
- Early retirement: complementarity problem associated with a parabolic integro-differential operator.
- The mathematical models are solved by using appropriate numerical methods.
- The results are compared with the ones obtained applying Monte Carlo simulation.
Thank you for your attention!